Multiple Zeros of Analytic Systems

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Objective

In [Dayton-Zeng, ISSAC 2005] we calculated the multiplicity of the analytic system

$$\begin{aligned} 1-\cos(x^2) &= 0 \\ \sin(y)+x^2e^{x+y} &= 0 \end{aligned}$$

at $\hat{\mathbf{x}} = (0,0)$ as $m = 4$. In this talk I would like to expand on this calculation.

Notation: By analytic system I mean that $f_1, f_2, \dots, f_n$ are complex valued functions in n-variables $x_1, \dots, x_n$ holomorphic in an open set $\mathcal{U}$ of $\mathbb{C}^n$. If $f_1, \dots, f_n$ are actually polynomials then we will call this a polynomial system.

Write $F = [f_1, \dots, f_n]^\top$. We will be interested in isolated zeros $\hat{\mathbf{x}} \in \mathcal{U}$ of the square system $F = \mathbf{0}$.

Multiplicity via DZ algorithm

(See [Dayton-Zeng, ISSAC 2005]). Let

$$\partial_{\mathbf{x}^{\mathbf{j}}} \equiv \partial_{x_1^{j_1} \cdots x_s^{j_n}} \equiv \frac{1}{j_1! \cdots j_n!} \frac{\partial^{j_1 + \cdots + j_n}}{\partial x_1^{j_1} \cdots \partial x_n^{j_s}}.$$

These partial derivatives give the coefficients of the Taylor series.

Let $\hat{\mathbf{x}}$ be an isolated zero of F. The *local array*, $L(k, \hat{\mathbf{x}})$ of, degree k at $\hat{\mathbf{x}}$ is the $n\binom{k+n-1}{n} \times \binom{k+n}{m}$ matrix with columns indexed by the monomials $\mathbf{x}^{\mathbf{j}}$ of degree $\leq k$ in degree lexicographical order. The rows are indexed by $\mathbf{x}^{\mathbf{i}} f_\alpha$ and grouped by degree of $\mathbf{x}^{\mathbf{i}}$. *The entry in the row indexed by $\mathbf{x}^{\mathbf{i}} f_\alpha$ and column indexed by $\mathbf{x}^{\mathbf{j}}$ is* $\partial_{\mathbf{x}^{\mathbf{j}}}((\mathbf{x}^{\mathbf{i}} - \hat{\mathbf{x}}^{\mathbf{i}}) f_\alpha)(\hat{\mathbf{x}})$.

$x_1 - x_2 + x_1^2, x_1 - x_2 + x_2^2$		1	x_1	x_2	x_1^2	x_1x_2	x_2^2	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3
	f_1	0	1	−1	1	0	0	0	0	0	0
$L(1, \hat{x})$	f_2	0	1	−1	0	0	1	0	0	0	0
	$x_1 f_1$	0	0	0	1	−1	0	1	0	0	0
	$x_1 f_2$	0	0	0	1	−1	0	0	0	1	0
	$x_2 f_1$	0	0	0	0	1	−1	0	1	0	0
$L(2, \hat{x})$	$x_2 f_2$	0	0	0	0	1	−1	0	0	0	1
	$x_1^2 f_1$	0	0	0	0	0	0	1	−1	0	0
	$x_1^2 f_2$	0	0	0	0	0	0	1	−1	0	0
	$x_1x_2 f_1$	0	0	0	0	0	0	0	1	−1	0
	$x_1x_2 f_2$	0	0	0	0	0	0	0	1	−1	0
	$x_2^2 f_1$	0	0	0	0	0	0	0	0	1	−1
$L(3, \hat{x})$	$x_2^2 f_2$	0	0	0	0	0	0	0	0	1	−1

DZ Multiplicity (continued)

In the exact polynomial case for an isolated zero $\hat{\mathbf{x}}$ the dimensions of the nullspaces of $L(k, \hat{\mathbf{x}})$, $\mathrm{nullity}(L(k, \hat{\mathbf{x}}))$, stabilize at some integer m as k increases. It is known that once $\mathrm{nullity}(L(k, \hat{\mathbf{x}})) = \mathrm{nullity}(L(k+1, \hat{\mathbf{x}}))$ then these nullities have stabilized and the common value m is the *multiplicity* of $\hat{\mathbf{x}}$, as originally calculated for polynomial systems by Macaulay in 1915.

The next lemma follows from standard results on complex analytic varieties.

Lemma

If $\hat{\mathbf{x}}$ is an exact isolated zero of an analytic system F, then $\mathrm{nullity}(L(k, \hat{\mathbf{x}}))$ *will stabilize at some finite integer k.*

Thus the DZ algorithm will give a *multiplicity* for analytic systems. We will argue below it is the correct one, at least in the exact case. Further, we will discuss the situation in the inexact case.

A simple Example

Let

$$F = \begin{bmatrix} \sin(x) - y \\ x - sin(y) \end{bmatrix}, \quad \hat{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We calculate $L(3, \hat{\mathbf{x}})$, a 12×10 matrix, and apply the reverse RREF (i.e. starting from the bottom right) to get the following matrix (zero rows dropped, a row of indices added).

$$\begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We see that all the columns with cubic indices are independent, so L has stabilized, moreover the non-pivot columns are indexed by $1, x, x^2$ so the nullity is 3. Thus this system has multiplicity 3 at the origin.

Numerical Computation

In general these local matrices are large and used with numerical data for either the functions f_i and/or the isolated zero $\hat{\mathbf{x}}$. Thus a numerical method should be used for calculating the nullity of these matrices. For example, one could use the [Li-Zeng, 2005] rank revealing method, or SVD setting singular values below a given tolerance to zero.

Later it will be useful to use a reverse RREF as above. The following Aproximate Reverse Reduced Row Echelon Form (ARRREF) for matrix L with tolerance ε was given in [Dayton, SNC 07].

- Find row space A of L by SVD, setting singular values less than ε to zero.
- Decide if the i^{th} column of A from the right is approxi-independent from the columns to the right of it, with tolerance ε, if so, that column is a *pivot* column, $i = 1, 2, \ldots, n$, (n is number of columns).
- Form the matrix B of the pivot columns, if B is square it is invertible, return ARRREF $= B^{-1}A$ otherwise return FAIL.

Using DZ with any of these methods the multiplicity is stable under small pertubations of F or $\hat{\mathbf{x}}$.

Pertubation Multiplicity

Let F be an analytic system with isolated zero at $\hat{\mathbf{x}}$ having DZ-multiplicity m. Let $G = [g_1, \ldots, g_n]$ be the system with g_i the Taylor series of f_i about $\hat{\mathbf{x}}$ of order k where k is large enough and $r > 0$ is small enough so that

- the nullity of the local array of F, $L(k, \hat{\mathbf{x}})$ has stabilized
- $\|F(\mathbf{x}) - G(\mathbf{x})\| < \|F(\mathbf{x})\|$ for all $\|\mathbf{x} - \hat{\mathbf{x}}\| = r$.

Then by construction the local arrays for F are exactly those of G so also the DZ multiplicity of G is m. If $\ell : \mathbb{C}^n \to \mathbb{C}^n$ is a small enough random linear function then

- The systems $F + \ell, G + \ell$ will, with probability 1, have no multiple zeros near $\hat{\mathbf{x}}$.
- $G + \ell$ will have m non-singular zeros $\hat{\mathbf{y}}_i$ with $\|\hat{\mathbf{y}}_i - \hat{\mathbf{x}}\| < r$,
- $\left\|\left(F(\mathbf{x}) + \ell(\mathbf{x})\right) - \left(G(\mathbf{x}) + \ell(\mathbf{x})\right)\right\| < \left\|F(\mathbf{x}) + \ell(\mathbf{x})\right\|$ for $\|\mathbf{x} - \hat{\mathbf{x}}\| = r$.

We apply

Rouche's Theorem

Theorem (Verschelde–Haegemans, 1994)

Let F, G be analytic systems and suppose $\|F - G\| < \|F\|$ on the boundary of a bounded domain $\mathcal{R}$. Then, modulo a small pertubation, the zeros of F, G inside $\mathcal{R}$ are connected by solution paths of the straight line homotopy $H(\mathbf{x}, t) = tG(\mathbf{x}) + (1 - t)F(\mathbf{x})$ lying in $\mathcal{R}$. In particular, F, G have the same number of zeros inside $\mathcal{R}$.

This theorem is applied where F, G are $F + \ell, G + \ell$ above. So the number of zeros of $F + \ell$ near $\hat{\mathbf{x}}$ is also m.

It is not necessary that the pertubation of the previous slide be restricted to adding a linear function, as long as the Jacobian of the pertubation becomes non-singular at $\hat{\mathbf{x}}$ the argument above will work.

Thus the DZ-multiplicity of an analytic system measures the number of zeros of a small pertubation of F near the multiple zero.

Multiplicity via Homotopy Continuation

Suppose one wants to find zeros of analytic system F using a polynomial start system G. Then one could use a straight line homotopy $H(\mathbf{x}, t) = tG + (1-t)F$. Assume F has a zero at $\hat{\mathbf{x}}$. Then for $\tilde{t}$ close to zero $H(\mathbf{x}, \tilde{t})$ is a pertubation of F. By above, if the multiplicity at $\hat{\mathbf{x}}$ is m then $H(\mathbf{x}, \tilde{t})$ will have m zeros near $\hat{\mathbf{x}}$ and any homotopy continuation path terminating at $\hat{\mathbf{x}}$ would have to go through one of them. Hence there are at most m distinct solution paths converging to $\hat{\mathbf{x}}$. Conversely the $G + \ell$ above would be a start system with m solution paths converging at $\hat{\mathbf{x}}$ by Rouche. Thus

Theorem

The multiplicity of a square analytic system F at $\hat{\mathbf{x}}$ is the largest number of distinct homotopy continuation paths from a polynomial start system converging to $\hat{\mathbf{x}}$.

Simple Example (Continued)

We take the Taylor series of $F = \begin{bmatrix} \sin(x) - y \\ x - sin(y) \end{bmatrix}$ center $\begin{bmatrix} 0.2 - 0.3i \\ -.1 + .3i \end{bmatrix}$ getting $G = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$ where

$$\begin{aligned} g1 = &- .000038 - .000201i + (.998484 + .001488i)x - y \\ &+ (.007687 + .001600i)x^2 + (-.17075 - .010083i)x^3 \\ g2 = &- .000106 + x + (1.00345 - .001213i)y \\ &+ (.004386 - .002998i)y^2 + (.173352 + .005067i)y^3 \end{aligned}$$

Zeros of $G = 0$ via PHCpack are approximately

$$\hat{\mathbf{y}}_1 = \begin{bmatrix} .03112 + .09899i \\ .03096 + .09885i \end{bmatrix}, \hat{\mathbf{y}}_2 = \begin{bmatrix} -.07460 - .07537i \\ -.07459 - .07535i \end{bmatrix}, \hat{\mathbf{y}}_3 = \begin{bmatrix} .05381 - .01061i \\ .05371 - .01071i \end{bmatrix}$$

Homotopy continuation from these starting values gives

$$\hat{\mathbf{z}}_1 = \begin{bmatrix} -.00002 + .00263i \\ -.00002 + .00263i \end{bmatrix}, \hat{\mathbf{z}}_2 = \begin{bmatrix} -.00212 - .00177i \\ -.00212 - .00177i \end{bmatrix}, \hat{\mathbf{z}}_3 = \begin{bmatrix} .00325 - .00127i \\ .00325 - .00127i \end{bmatrix}$$

indicating multiplicity 3.

The Griewank-Osborne Analytic Example

The following example was studied in [Griewank and Osborne, 1983].

$$
\begin{aligned}
0.5x^2\cos(10x) + x\sin(10y) + 10x\sinh(z) &= 0\\
\cos(5x) + \sin(5y) - 1 &= 0\\
\cosh(5x) + 5\sinh(z) - 1 &= 0
\end{aligned}
$$

There is a zero at $(0, \frac{\pi}{5}, 0)$. We attempt to find the multiplicity at approximate zero $\hat{\mathbf{x}} = (0, 0.62832, 0)$ using the DZ algorithm by calculating $L(5, \hat{\mathbf{x}})$. This is a 105×56 matrix with approximate entries. Therefore we use an approximate method to calculate the rank, i.e. SVD with tolerance 10^{-6} on the singular values. The approxi-dimension of the nullspace is then 3. It is clear this nullity has stabilized so we conclude a multiplicity of 3.

Griewank-Osborne (continued)

Starting from the zeros of an order 3 Taylor approximation of the system centered at a random point near $\hat{\mathbf{x}}$, homotopy continuation yields the following three essentially real (very small imaginary parts) points:

$$
\hat{\mathbf{z}}_1 = \begin{bmatrix} .123 * 10^{-4} \\ .62832 \\ .741 * 10^{-11} \end{bmatrix},\ \hat{\mathbf{z}}_2 = \begin{bmatrix} -.854 * 10^{-4} \\ .62832 \\ .885 * 10^{-11} \end{bmatrix},\ \hat{\mathbf{z}}_3 = \begin{bmatrix} .00995 \\ .62807 \\ -.000247 \end{bmatrix}
$$

The first two points are very close to $\hat{\mathbf{x}}$, but the second is farther off. Note $\|\hat{\mathbf{z}}_3 - \hat{\mathbf{x}}\| \approx .01$ which is small enough that one may consider these three points a cluster and accepting multiplicity 3 in the numerical sense as suggested by [Kahan, 1972] or more recently [Jonovitz-Freireich, Ronyai, Szanto, ISSAC 2006].

Numerical Local Ring Approach

In a recent paper [Dayton, SNC '07] I showed how one could further analyze the situation above using the DZ approach. The main observation is that, in the exact polynomial case, the rows of the local array, viewed as polynomials by multiplying entries by column indices, give a presentation of the local ring at $\hat{\mathbf{x}}$. In the present numeric case the ideal generated by the row space may not give a local ring (one maximal ideal - i.e. zero) and/or may not be entirely consistent. The following method often reveals points close to $\hat{\mathbf{x}}$.

NLR Method

- Find k so that $L(k, \hat{\mathbf{x}})$ has highest degree columns independent.
- Apply ARRREF to get approximate Gröbner basis.
- Use Möller-Stetter method with refinement step to obtain zeros.

Griewank-Osborne (concluded)

$$
\begin{aligned}
0.5x^2\cos(10x) + x\sin(10y) + 10x\sinh(z) &= 0\\
\cos(5x) + \sin(5y) - 1 &= 0\\
\cosh(5x) + 5\sinh(z) - 1 &= 0
\end{aligned}
$$

Applying the NLR method to the Griewank-Osborne example above at point $\hat{\mathbf{x}}$ one obtains two zeros essentially identical to $\hat{\mathbf{x}} = (0, .62832, 0)$ and a third which is extremely close to

$$
\hat{\mathbf{z}}_3 = \begin{bmatrix} .00995 \\ .62807 \\ -.000247 \end{bmatrix}
$$

This suggests that $\hat{\mathbf{z}}_3$ did not vary from $\hat{\mathbf{x}}$ due to numerical error in the homotopy continuation but is actually a separate zero of this exact system.

Refining $\hat{\mathbf{z}}_3$ using Newton's Method at a high precision confirms this. Therefore the exact zero $(0, \frac{\pi}{5}, 0)$ has multiplicity 2 with nearby simple zero $\hat{\mathbf{z}}_3$.

Selected References

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