

# Local Solutions of Analytic Systems by Homotopy

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July 22, 2007  
Applications of Computer Algebra 2007  
Oakland University

## Griewank-Osborne System

In my SNC 07 paper *Numerical Local Rings and Local Solution of Nonlinear Systems* [NLR] I analyzed an example of Griewank and Osborne (1983) on real Newton's method.

$$\begin{aligned}0.5x^2 \cos(10x) + x \sin(10y) + 10x \sinh(z) &= 0 \\ \cos(5x) + \sin(5y) - 1 &= 0 \\ \cosh(5x) + 5 \sinh(z) - 1 &= 0\end{aligned}$$

Griewank and Osborne noted the origin was a multiple zero, there was also a simple zero at approximately

$$(0.1159, 0.0328, -0.0345)$$

This partly explained the poor convergence of real Newton's method near the origin.

$$\begin{aligned}
 &0.5x^2 \cos(10x) + x \sin(10y) + 10x \sinh(z) \\
 &\cos(5x) + \sin(5y) - 1 \\
 &\cosh(5x) + 5 \sinh(z) - 1
 \end{aligned}$$

## Griewank-Osborne System

Using NLR the origin is a double zero and I found 4 additional simple zeros near the origin:

$$\begin{aligned}
 &(0.0653 \mp 0.2606i, -0.1534 \mp 0.1401i, 0.1440 \pm 0.0635) \\
 &(-0.1297 \pm 0.0818i, 0.0263 \mp 0.0507i, -0.0235 \pm 0.0553)
 \end{aligned}$$

Thus the one could say that there was a *numerical multiplicity* of 7 near the origin.

**Question 1:** Are there any other zeros near the origin? Does my NLR method miss anything?

$$\begin{aligned}
 &0.5x^2 \cos(10x) + x \sin(10y) + 10x \sinh(z) \\
 &\cos(5x) + \sin(5y) - 1 \\
 &\cosh(5x) + 5 \sinh(z) - 1
 \end{aligned}$$

## Griewank-Osborne System

To answer this question note

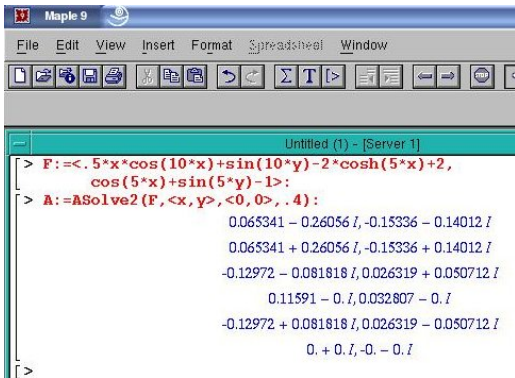
- ▶  $x$  can be factored out of the first equation losing multiplicity but not solutions.
- ▶ The variable  $z$  can be eliminated by using the third equation.

We get a new system GO2 with zeros giving  $x, y$  coordinates of solutions of GO.

$$\begin{aligned}
 &0.5x \cos(10x) + \sin(10y) - 2 \cosh(5x) + 2 = 0 \\
 &\cos(5x) + \sin(5y) - 1 = 0
 \end{aligned}$$

# Maple Solution

Use ASolve2.



```
Maple 9
File Edit View Insert Format Spreadsheet Window
[Icons]
Untitled (1) - [Server 1]
[> F := .5*x*cos(10*x)+sin(10*y)-2*cosh(5*x)+2,
    cos(5*x)+sin(5*y)-1]:
[> A:=ASolve2(F, <x, y>, <0, 0>, .4):
    0.065341 - 0.26056 I, -0.15336 - 0.14012 I
    0.065341 + 0.26056 I, -0.15336 + 0.14012 I
    -0.12972 - 0.081818 I, 0.026319 + 0.050712 I
    0.11591 - 0. I, 0.032807 - 0. I
    -0.12972 + 0.081818 I, 0.026319 - 0.050712 I
    0. + 0. I, -0. - 0. I
[>
```

$$0.5x^2 \cos(10x) + x \sin(10y) + 10x \sinh(z) \\ \cos(5x) + \sin(5y) - 1 \\ \cosh(5x) + 5 \sinh(z) - 1$$

## Griewank-Osborne System

continued

By observation there are further zeros at  $(0, \pm \frac{\pi}{5}, 0)$ , note  $\frac{\pi}{5} \approx 0.6283$ , and NLR verified these are double zeros with nearby simple real zeros  $(0.0099, \pm 0.6281, -0.0002)$

**Question 2:** Are there other zeros within Euclidean distance 1 of the origin besides the 13 zeros already identified.

# Mathematica Solution

(Mathematica provided by Oakton Community College)

```
usr2:MathematicaWork\G062007.nb
File Edit Window Help

f1 = .5*x^2*Coe[10*x] + x*Sin[10*y] + 10*x*Sinh[z]
0.5 x^2 Cos[10 x] + x Sin[10 y] + 10 x Sinh[z]

f2 = Coe[5*x] + Sin[5*y] - 1
-1 + Cos[5 x] + Sin[5 y]

f3 = Cosh[5*x] + 5*Sinh[z] - 1
-1 + Cosh[5 x] + 5 Sinh[z]

solve3[f1, f2, f3, {0, 0, 0, 1}]
{{x - 6.86925*10^-8 - 4.17715*10^-8 i, y - 0.628319 + 5.11532*10^-24 i, z - 1.34814*10^-17 + 5.11532*10^-24 i},
{x - 0.00995054 - 1.19678*10^-13 i, y - 0.628071 + 3.50838*10^-13 i, z - 0.000247584 + 3.50723*10^-13 i},
{x - 0.115914 + 2.25574*10^-17 i, y - 0.0328072 - 1.03437*10^-17 i, z - 0.034534 - 1.09486*10^-17 i},
{x - 6.71259*10^-10 + 7.35888*10^-8 i, y - 0.628319 + 8.41735*10^-25 i, z - 4.11419*10^-18 + 8.41735*10^-25 i},
{x - 6.41776*10^-9 - 9.27764*10^-10 i, y - 1.86152*10^-17 + 2.06005*10^-24 i, z - 3.5893*10^-18 - 2.06005*10^-24 i},
{x - 0.00995054 - 8.49661*10^-18 i, y - 0.628566 + 2.29269*10^-18 i, z - 0.000247584 + 2.29139*10^-18 i},
{x - 0.129722 + 0.0818175 i, y - 0.0263189 - 0.0507123 i, z - 0.0235007 + 0.0553074 i},
{x - 0.0653412 + 0.260561 i, y - 0.153358 - 0.14012 i, z - 0.143983 - 0.0635194 i}}
```

Two more applications of solve3 give all 13 known zeros and only the expected 13 solutions.

## One Variable Solver (MATLAB)

See also T.-Y. Li, 1983

- ▶ Equation GO1 defined by input file afun5.m

```
function [y,dy]=afun5(x)
f= .5*x^2*cos(10*x)-2*x*cosh(5*x)+2*x;
df = 1.0*x*cos(10*x)-5.0*x^2*sin(10*x)-2*cosh(5*x)
-10*x*sinh(5*x)+2;
```

- ▶ S = asolve1('afun5',0,1)

```
S =
-0.1774 + 0.3483i
 0.0006 + 0.0013i
-0.7401 + 0.5228i
-0.7401 - 0.5228i
 0.4721 + 0.4302i
-0.1774 - 0.3483i
 0.0001 - 0.0013i
 0.0196 + 0.0000i
 0.4721 - 0.4302i
```

## Rouche's Theorem

### Theorem

Let  $\mathcal{R}$  be a bounded region with boundary  $\partial\mathcal{R}$  with  $\mathcal{R} \cup \partial\mathcal{R}$  contained in an open set  $\mathcal{U} \subseteq \mathbb{C}^n$  and  $F, G : \mathcal{U} \rightarrow \mathbb{C}^n$  be holomorphic. Let  $\|\cdot\|$  be any norm on  $\mathbb{C}^n$  and suppose  $\|G\| > \|G - F\|$  on  $\partial\mathcal{R}$ . Then  $F, G$  have the same number of zeros counted by multiplicity in  $\mathcal{R}$ .

### Proof by Verschelde, Haegemans, 1994.

Use homotopy continuation. Consider the straight line homotopy

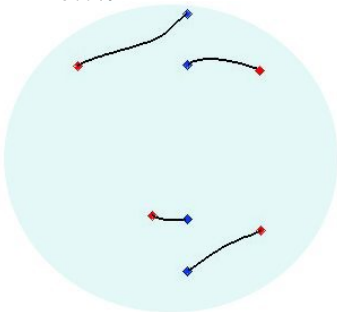
$$H(\mathbf{z}, t) = tF(\mathbf{z}) + (1 - t)G(\mathbf{z})$$

There are two steps

- ▶ Slightly perturb  $F, G$  and  $H$  so there are no singularities on the solution paths for  $0 \leq t \leq 1$ .
- ▶ Note the condition  $\|G\| > \|G - F\|$  implies no solution paths cross the boundary  $\partial\mathcal{R}$  when  $0 \leq t \leq 1$ .

## Rouche's Theorem – Proof concluded

Thus the solution paths give a 1-1 correspondance between zeros of  $F$  and  $G$ . In fact the zeros of  $G$  can almost always be obtained by Homotopy continuation using as a start system the zeros of  $F$  inside  $\mathcal{R}$ .



Red = zeros of  $F$   
Blue = zeros of  $G$

## Applications

### Corollary

Consider the 1-norm  $\|(\mathbf{z})\|_1 = \max(|z_1|, \dots, |z_n|)$  and let  $\hat{\mathbf{z}} \in \mathbb{C}^n$ . Suppose  $\mathcal{R} = \{\mathbf{z} \in \mathbb{C}^n \mid \|\mathbf{z} - \hat{\mathbf{z}}\|_1 < r\} \subseteq \mathcal{U}$  and  $G : \mathcal{U} \rightarrow \mathbb{C}^n$  is holomorphic and has no zeros on  $\{\mathbf{z} \in \mathbb{C}^n \mid \|\mathbf{z} - \hat{\mathbf{z}}\|_1 = r\}$ . Then there exists a polynomial system  $F$  such that the zeros of  $F$  inside  $\mathcal{R}$  give a start system for finding all zeros of  $G$  inside  $\mathcal{R}$ .

**Proof:** There will be a Taylor approximation of  $G$  satisfying  $\|G\|_1 > \|G - F\|_1$  on  $\partial\mathcal{R}$ .

## Application to DZ algorithm

B.H. Dayton, Z. Zeng, ISSAC 2005

The multiplicity of an isolated zero  $\hat{\mathbf{z}}$  of a holomorphic system  $F$  may be defined topologically as the Brouwer degree (argument) of  $\frac{F(\mathbf{z})}{\|F(\mathbf{z})\|}$  on the boundary of a small ball about  $\hat{\mathbf{z}}$ . This is the number of zeros near  $\hat{\mathbf{z}}$  obtained by slightly perturbing  $F$ .

### Theorem

If  $F$  is a holomorphic system with isolated zero  $\hat{\mathbf{z}}$  then the DZ algorithm gives the multiplicity defined above.

**Example:** ASolve2 (Maple) applied to the system in DZ  $f_1 = 1 - \cos(x^2)$ ,  $f_2 = \sin(y) + x^2 e^{x+y}$  gives 4 zeros close to the origin, agreeing with our earlier multiplicity calculation.

## Impracticality of Rouche

- ▶ A suitable Taylor series may be of too high an order to be useful. For example if  $\mathcal{R} = \{\mathbf{z} \mid \|\mathbf{z}\| < 1\}$  then the GO system could require a Taylor approximation of order 40 or more.
- ▶  $\|G\| > \|G - F\|$  says the relative error of approximating  $G$  by  $F$  on  $\partial\mathcal{R}$  is less than 100%. This may be a weak condition if  $\min\{\|G(\mathbf{z})\| \mid \mathbf{z} \in \partial\mathcal{R}\}$  is large on  $\partial\mathcal{R}$ , but for analytic functions this may be small for most regions, hence the condition  $\|G\| > \|G - F\|$  may be hard to satisfy and/or verify.

But, even when  $\|G\| > \|G - F\|$  fails, we get useful information.

## Rouche restated

For  $F, G$  holomorphic on an open set containing the closure of a region  $\mathcal{R}$ , and let

- ▶  $N_F$ , (resp  $N_G$ ) be the number of zeros of  $F$  (resp  $G$ ) in  $\mathcal{R}$ .
- ▶  $N_H$  be the cardinality of

$$\begin{aligned} & \left\{ \mathbf{z} \in \partial\mathcal{R} \mid tF(\mathbf{z}) + (1-t)G(\mathbf{z}) = 0, \text{ some } 0 < t < 1 \right\} \\ & = \left\{ \mathbf{z} \in \partial\mathcal{R} \mid \frac{F(\mathbf{z})}{\|F(\mathbf{z})\|} = -\frac{G(\mathbf{z})}{\|G(\mathbf{z})\|} \right\} \end{aligned}$$

### Theorem

Assume no singularities of the homotopy on  $\partial\mathcal{R}$ , with notation as above,  $|N_G - N_F| \leq N_H$

## Rouche restated again

### Theorem

Assume there are no singularities of  $H(\mathbf{z}, t) = tF(\mathbf{z}) + (1 - t)G(\mathbf{z})$  on  $\partial\mathcal{R}$ .

- ▶ With notation as above,  $|N_G - N_F| \leq N_H$ .
- ▶ There are at least  $N_G - N_H$  zeros of  $G$  in  $\mathcal{R}$  which can be obtained from homotopy paths starting from zeros of  $F$  in  $\mathcal{R}$ .

### Experimental Observation

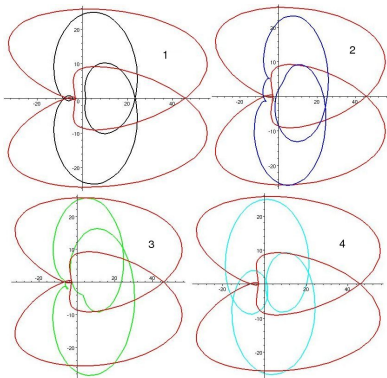
If  $F$  is a sufficiently generic approximation of  $G$  on  $\partial\mathcal{R}$  then  $N_H$  tends to be small.

In particular one needs to avoid sparse polynomial approximations and Taylor series with real centers.

If approximations are random and loose enough, different zeros will be found. Use slightly overdetermined least square interpolations.

## Example of least square approximations

Analytic equation of one variable



The red curve is the graph original analytic function on a circle about the origin.

In picture 1 the approximation is the Taylor series on that circle. In the other pictures a least square approximation with random center is given.

## ASOLVE algorithm

- ▶ **INPUT:** System  $G = \mathbf{0}$ , center  $\mathbf{z}_0$  and radius  $r$ .
- ▶ Randomly shift center to nearby  $\mathbf{z}_1$ , increase  $r$  to  $r_1 = r + \|\mathbf{z}_1 - \mathbf{z}_0\|_2$
- ▶ Calculate the least square interpolation polynomial  $F$  of  $G$  of fixed degree using small specific set of points randomly rotated.
- ▶ Find the solution set  $X$  of the polynomial system  $F = \mathbf{0}$ .
- ▶ With  $F$  and solutions  $\{\mathbf{z} \in X \mid \|\mathbf{z} - \mathbf{z}_1\|_2 < r_1\}$  as a start system use homotopy continuation to find solutions of  $G = \mathbf{0}$ .
- ▶ **OUTPUT:** Return those solutions  $\mathbf{z}$  satisfying  $\|\mathbf{z} - \mathbf{z}_0\|_2 < r$ .

This algorithm should be run several times.

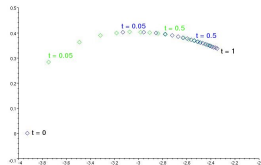
## Implementation Notes

- ▶ **Interpolation:** Use random rotation of vertex set from subdivision of minimal triangulation of  $S^{2n-1} \subseteq \mathbb{C}^n$ .



First subdivision of regular 2-sphere triangulation. (14 vertices)

- ▶ **Homotopies:** The polynomial function tends to dominate until  $t = 0$  so there is often a big jump at the end. Adaptive tracking is needed. Tracking works better with the homotopy  $H(\mathbf{z}, t) = t^2 F(\mathbf{z}) + (1 - t^2) G(\mathbf{z})$ .



Blue:  $H = tF + (1 - t)G$

Green:  $H = t^2 F + (1 - t^2)G$

## Implementation and Examples








- ▶ **1 variable:** MATLAB, MAPLE, Mathematica
- ▶ **2 variables:** MAPLE, Mathematica
- ▶ **3 variables:** Mathematica

Software available from author, [bhdayton@neiu.edu](mailto:bhdayton@neiu.edu)

### Additional Examples

- ▶  $\tan x \pm \frac{x^3}{1-x^2} = 0, r=1$  (Domain  $-\frac{\pi}{2} < x < \frac{\pi}{2}$ )
- ▶ J-P. Merlet test problem, [web]  
 $\sin(x)^2 * \cos(x) - .7071067810 * \sin(x)^2 - 1.366025404 * \sin(x) * \cos(x)$   
 $+ .9659258263 * \sin(x) - .3061862179 + .4330127020 * \cos(x) = 0, r=3.2$
- ▶  $2\text{cn}(x) + 3\text{sn}(x) = 0, r=4$  where  $\text{cn}, \text{sn}$  are Jacobi Elliptic Functions  $\kappa = .7$
- ▶  $\begin{cases} \sin(x^2 - y) = 0 \\ \sin(x^2) - \sin(y) = 0 \end{cases}, r=1, \text{ One dimensional component } y=x^2.$

## References

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